Path integral for minisuperspaces and its relation with non equivalent canonical quantizations

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ABSTRACT

The relation between a recently proposed path integral for minisuperspaces and different canonical quantizations is established. The step of the procedure where a choice between non equivalent theories is made is identified. Coordinates avoiding such a choice are found for a class of homogeneous cosmologies.

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A central problem of quantum cosmology is the absence of a true time in the formalism for the gravitational field, which, in turn, includes a non physical time parameter τ [1,2,3]. A possible solution for cosmological models can be the identification of a physical time in terms of a subset of the canonical variables describing the system under consideration [4,5,6,7]. Within this framework, in a recent book [8] we proposed a way to identify a time and obtain a consistent path integral quantization by establishing a correspondence between the action functional $S[q^i, p_i, N]$ of homogeneous cosmologies and the action $\mathcal{S}[Q^i, P_i, N]$ of an ordinary gauge system [9,10,11]. Canonical gauges imposed on the gauge system define a global time for a given minisuperspace, thus yielding a quantization with a clear distinction between time and true degrees os freedom.

A cosmological model admits our treatment if its Hamiltonian constraint $\mathcal{H}$ allows for the existence of coordinates $\tilde{q}^i(q^i, p_i)$ such that one of them is a global time, that is, $\{\tilde{q}^0, \mathcal{H}\} > 0$ [4], and such that the Hamilton–Jacobi equation associated with $\mathcal{H}$ (or with the scaled constraint H) is solvable. If these conditions are fulfilled, two succesive canonical transformations $(\tilde{q}^i, \tilde{p}_i) \to (\overline{Q}^i, \overline{P}_i) \to (Q^i, P_i) = (Q^\mu, Q^0, P_\mu, P_0)$ are performed, which match the Hamiltonian constraint $\mathcal{H}$ with the new momentum P_0; then the variables (Q^i, P_i) describe an ordinary gauge system, and a canonical gauge like $\chi \sim Q^0 - T(\tau) = 0$, which does not generate Gribov copies [6,12], defines a time in terms of the variables $(\tilde{q}^i, \tilde{p}_i)$ of the model. After gauge fixation and on the constraint surface $\mathcal{H} = P_0 = 0$ the propagator takes the form [12,8]

$$\left\langle \tilde{q}_2^i | \tilde{q}_1^i \right\rangle = \int DQ^\mu DP_\mu \exp\left(i \int_{\tau_1}^{\tau_2} [P_\mu dQ^\mu - h(Q^\mu, P_\mu, \tau) d\tau] \right), \tag{1}$$

where h is the (true) Hamiltonian for the reduced system described by the observables $(Q^\mu(\tilde{q}^i, \tilde{p}_i, \tau), P_\mu(\tilde{q}^i, \tilde{p}_i, \tau))$.

However, in the case of certain cosmological models there are some subtleties which have not been analysed in detail in our previous works, but which deserve a thorough discussion, as they involve a definite choice between non equivalent quantizations associated

to classically equivalent formulations; this choice is not of the usual form associated to different operator orderings, i.e. one cannot pass from one formulation to another one by changing the operator ordering. In some of our works the problem was avoided, but this was achieved at the price of obtaining a propagator which could not be explicitly solved [13,14].

The purpose of the present letter is thus to clarify these points and improve the proposed procedure. First, we shall establish which is the canonical quantization which corresponds to our path integral formulation, and at which stage of our procedure we make a definite choice. Within this context, we shall analyse the positive and negative aspects of different ways of performing our deparametrization and quantization program. Finally, we shall propose a coordinate change making unnecesary the choice between two non equivalent quantum theories in the case of a class of minisuperspaces including, among others, some string cosmological models which have recently received attention in the literature [15,16,17,13,8], as well as some anisotropic universes. Also, we shall show that our procedure is not restricted to models which admit an intrinsic time.

The discussion can be illustrated by the first model considered in Ref. [12], which is a flat Friedmann–Robertson–Walker universe with a massless scalar field ϕ and positive cosmological constant Λ, its scaled Hamiltonian constraint given by $H = -p_\Omega^2 + p_\phi^2 + \Lambda e^{6\Omega} = 0$, ($\Omega \sim \ln a$, a the scale factor). However, we shall begin by considering a generic constraint of the form

$$H = -\tilde{p}_1^2 + \tilde{p}_2^2 + Ae^{(a\tilde{q}^1 + b\tilde{q}^2)} = 0 \tag{2}$$

with $a \neq b$. This kind of constraint is common in dilatonic models (that is, cosmologies coming from the low energy limit of bosonic string theory), and also includes isotropic and anisotropic relativistic models, like the Kantowski–Sachs universe, or even the Taub universe after a suitable canonical transformation [14]; the latter is an example in which $\tilde{q}^0 = \tilde{q}^0(q^i, p_i)$, so that the time is extrinsic [18,19]. It is easy to show that a coordinate

change exists such that this constraint can be put in the form

$$H = -p_x^2 + p_y^2 + \zeta e^{2x} = 0, \tag{3}$$

with $sgn(\zeta) = sgn(A/(a^2 - b^2))$; clearly, for $\zeta > 0$ we have a system analogous to the first studied in Ref. [12]. In the case $\zeta < 0$ the momentum p_y does not vanish on the constraint surface; hence $\{p_y, H\} \neq 0$ and the coordinate y is a global time . For $\zeta > 0$, instead, we have $\{p_x, H\} \neq 0$ and a global time is the coordinate x.

The case involving the aforementioned subtlety corresponds to $\zeta > 0$. The straightforward application of our deparametrization and path integral quantization procedure to the system described by the Hamiltonian (3) with $\zeta > 0$ yields, after imposing a canonical gauge condition, a global time $t = -x\, sgn(p_x) \equiv -\eta x$ and the reduced phase space path integral [12]

$$\langle y_2, x_2 | y_1, x_1 \rangle = \int DQDP \exp\left(i \int_{T_1}^{T_2} \left[PdQ + \eta\sqrt{P^2 + \zeta e^{2T}} dT \right] \right), \tag{4}$$

where the endpoints are $T_1 = x_1$ and $T_2 = x_2$, and the paths go from $Q_1 = y_1$ to $Q_2 = y_2$.

To establish which is the canonical quantization corresponding to this path integral we can note the following: from the reduced action yielding after gauge fixation we can read the time-depending true Hamiltonian $h = -\eta\sqrt{p_y^2 + \zeta e^{2x}}$. Because in this case $x = -\eta t$, then $h = -\tilde{p}_0 = -p_x$ and we have

$$-p_x + \eta\sqrt{p_y^2 + \zeta e^{2x}} = 0,$$

thus obtaining two constraints, namely $K^+ = 0$ and $K^- = 0$, each one linear in the momentum p_x. These two constraints together are classically equivalent to the original Hamiltonian constraint $H = 0$, which is quadratic in all the momenta; that is, classical dynamics take place in one of two sheets in which the constraint surface splits. But at the quantum level this equivalence does no more hold: a function in the kernel of the operator $\hat{K}^+$ or $\hat{K}^-$ is not annihilated by the operator $\hat{H}$, but by $\hat{H}$ plus terms corresponding

to a commutator between $\hat{p}_x$ and the square-root true Hamiltonian resulting from its time-depending potential (see below). It must be emphasized that these terms cannot be eliminated by any operator ordering; thus the choice of limiting procedure in the skeletonization of the path integral, which determines a particular ordering, is not relevant for our analysis. Imposing the operator form of the original Hamiltonian constraint on a wave function yields the usual Wheeler–DeWitt equation which is of second order in $\partial/\partial x$. Instead, splitting the constraint into two disjoint sheets yields a canonical quantization consisting in two equations of first order in $\partial/\partial x$:

$$i\frac{\partial}{\partial x}\Psi = \pm\left(-\frac{\partial^2}{\partial y^2} + \zeta e^{2x}\right)^{1/2}\Psi.$$

In this case we have a pair of Hilbert spaces, each one with its corresponding Schrödinger equation. We can say that the Schrödinger quantization preserves the topology of the constraint surface, that is, the splitting of the classical solutions into two disjoint subsets has its quantum version in the splitting of the theory into two Hilbert spaces.

Therefore, our path integral quantization is not in correspondence with the solutions of a Wheeler–DeWitt equation, but with those of a pair of Schrödinger equations, one for each sheet of the constraint surface (we want to insist on the point that operator ordering plays a secondary role within this context). Though the Wheeler–DeWitt equation is the most common choice for the canonical quantization of minisuperspaces, a time-depending potential in the Hamiltonian constraint makes difficult the interpretation of the resulting wave function in tems of a conserved positive-definite inner product (this cannot be avoided by defining the time as $\pm y$, because when $\zeta > 0$ we do not have $\{y, H\} \neq 0$ everywhere; such a wrong choice would lead to a reduced Hamiltonian which is not self-adjoint). The Schrödinger quantization, instead, allows to define a conserved inner product for each subset of solutions associated to each sheet of the constraint; this inner product is defined

by fixing the time in the integration:

$$(\Psi_1|\Psi_2) = \int dq\, \delta(t - t_0)\Delta \Psi_1^* \Psi_2,$$

where Δ is a determinant making the integral independent of the time choice (if one of the coordinates q explicitly appearing in the wave function is itself the time, then $\Delta = 1$). Hence our path integral formulation is associated to the canonical quantization which allows for a clear probability interpretation, and which, in a sense, reproduces the classical geometry of the constraint surface. Our procedure could be understood as introducing the appropriate quantum corrections to the Hamiltonian constraint H [20], which are given by the commutator mentioned above, and whose general form is $\left[\sqrt{\sum(\hat{\tilde{p}}_r)^2 + V(\hat{\tilde{q}}^i)}, \hat{\tilde{p}}_0\right]$ (where $r \neq 0$, and V stands for the potential in the scaled Hamiltonian constraint H).

The choice between both formalisms is made when we solve the Hamilton–Jacobi equation $H\left(\tilde{q}^i, \partial W/\partial \tilde{q}^i\right) = E$: because of the quadratic form of the constraint, this is a first order non linear equation, but to explicitly obtain its solution W one integrates two equations which are linear in the derivative respect to the coordinate identified as $\pm$ the time,

$$\frac{\partial W}{\partial \tilde{q}^0} = \pm \left(\sum_r \left(\frac{\partial W}{\partial \tilde{q}^r}\right)^2 + V(\tilde{q}^i) - E \right)^{1/2}.$$

At the level of the path integral this is reflected in the fact that the reduced Hamiltonian h in (1) is equal to $\partial f/\partial \tau$, where f is chosen to ensure that the path integral in the new variables effectively corresponds to the transition amplitude $\langle \tilde{q}_2^i | \tilde{q}_1^i \rangle$. This is achieved if the end point terms [11,12]

$$B = \left[\overline{Q}^i \overline{P}_i - W(\tilde{q}^i, \overline{P}_i) + Q^\mu P_\mu - f(\overline{Q}^\mu, P_\mu, \tau) \right]_{\tau_1}^{\tau_2}$$

associated to the two succesive canonical transformations vanish on the constraint surface $H = 0$ and in a gauge associated to a global time $t(\tilde{q}^i)$; this yields two disjoint theories, one for each reduced Hamiltonian determined by each sign of W .

In some of our works [13,14] we proposed a change to null coordinates leading to the following form of the constraint:

$$p_x p_y + C = 0. \tag{5}$$

The obtention of a constant 'potential' then seems to solve the problem associated to the existence of non equivalent forms of writing the constraint. The application of our method starting from this constraint leads to an action including a true Hamiltonian which is time-independent. However, the resulting propagator for the physical degree of freedom

$$\langle y_2, x_2 | y_1, x_1 \rangle = \int DQDP \exp\left(i \int_{T_1}^{T_2} \left[PdQ - \frac{\eta}{P} dT \right] \right) \tag{6}$$

(where the paths go from $Q_1 = y_1$ to $Q_2 = y_2$ and the endpoints are $T_1 = x_1,\ T_2 = x_2$) has the unsatisfactory feature that the functional integration over P cannot be effectively performed, even to find its infinitesimal form, because of the dependence P^{-1} of the reduced Hamiltonian.

For the class of models studied here we can propose a coordinate choice with the following desirable properties: 1) The necessity of deciding between inequivalent quantum theories yielding from different forms of writing the classical constraint is avoided. 2) The functional integration can be explicitly performed. Consider the constraint (2) and define

$$\begin{aligned} u &= \alpha \exp\left(\frac{a\tilde{q}^1 + b\tilde{q}^2}{2} \right) \cosh\left(\frac{b\tilde{q}^1 + a\tilde{q}^2}{2} \right) \\ v &= \alpha \exp\left(\frac{a\tilde{q}^1 + b\tilde{q}^2}{2} \right) \sinh\left(\frac{b\tilde{q}^1 + a\tilde{q}^2}{2} \right), \end{aligned} \tag{7}$$

with $\alpha = \sqrt{|A|}$. Because $a \neq b$ and $u \neq v$ these coordinates allow to write the constraint in the equivalent (scaled) form

$$H = -p_u^2 + p_v^2 + \eta m^2 = 0, \tag{8}$$

with $\eta = sgn(A)$ and $m^2 = 4/|a^2 - b^2|$. It is clear that commutators cannot appear now, or, in other words, the Wheeler–DeWitt equation is equivalent to two Schrödinger

equations. The time will be u or v depending on η; formally, this yields from a canonical gauge of the form $\chi \equiv \sqrt{P^2+m^2}Q^0 - T(\tau) = 0$. The application of our path integral procedure to this system is straightforward; on the constraint surface and after gauge fixation we obtain:

$$\langle u_2, v_2 | u_1, v_1 \rangle = \int DQ\, DP \exp\left(i \int_{T_1}^{T_2} \left[P dQ + \eta \sqrt{P^2+m^2} dT \right] \right), \tag{9}$$

where the endpoints are $T_1 = u_1(v_1)$ and $T_2 = u_2(v_2)$ and the paths go from $Q_1 = v_1(u_1)$ to $Q_2 = v_2(u_2)$ for $\eta = 1(-1)$. By skeletonizing the paths we obtain $N-1$ δ-functionals of the form $\delta(P_m - P_{m-1})$, and hence the functional integration reduces to an ordinary one:

$$\langle u_2, v_2 | u_1, v_1 \rangle = \int dP \exp\left(i \left[P(Q_2 - Q_1) + \eta \sqrt{P^2+m^2} (T_2 - T_1) \right] \right). \tag{10}$$

The double sign given by η corresponds to both possible sheets of the constraint surface where the evolution can take place.

Let us illustrate this coordinate choice with some simple dilatonic cosmologies (see [8] and references therein); consider the scaled constraint

$$H = -p_\Omega^2 + p_\phi^2 + 2ce^{6\Omega+\phi} = 0$$

which corresponds to a flat model with dilaton field ϕ. In the case $c < 0$ this constraint admits a change to the coordinates x and y yielding an expression like (3), with $t = \pm y$, but for $c > 0$ we have $t = \pm x$ and the problem of the time-depending potential appears. The change to (u, v), instead, solves this: for $c < 0$ we have $t = \pm v$, while for $c > 0$ we obtain $t = \pm u$. Note that in the case $c < 0$ (for which the dilaton ϕ is itself a globally good time as $p_\phi \neq 0$), we obtain $-\infty < t < \infty$ on boths sheets of the constraint determined by the sign of p_v; in the case $c > 0$ (which admits Ω as a global time), instead, we have that t goes from $-\infty$ to 0 on the sheet $p_u > 0$, and from 0 to ∞ on the sheet $p_u < 0$, with $t \to 0$ corresponding to the singularity $\Omega \to -\infty$. If we add a term representing the

inclusion of a non vanishing antisymmetric field $B_{\mu\nu}$ coming from the NS-NS sector of effective string theory, we have the constraint

$$H = -p_\Omega^2 + p_\phi^2 + 2ce^{6\Omega+\phi} + \lambda^2 e^{-2\phi} = 0$$

which in principle does not admit the proposed coordinate change. Moreover, in the case $c < 0$ the model does not admit an intrinsic time. However, we should recall that, because these models come from the low energy string theory, which makes sense in the limit $\phi \to -\infty$, then the $e^\phi \equiv V(\phi)$ factor in the first term of the potential verifies $V(\phi) = V'(\phi) \ll 1$ (this is clearly not the case with the term $e^{-2\phi}$), and we can replace $ce^\phi \to \overline{c}$. As a previous step we can then perform the canonical transformation introduced for the Taub universe in Refs. [7,14] to obtain a constraint with only one term in the potential: for both signs of $\overline{c}$ we can define the generator $f_1 = \pm|\lambda|e^{-\phi}\sinh s$ of a canonical transformation leading to

$$H = -p_\Omega^2 + p_s^2 + 2\overline{c}e^{6\Omega} = 0$$

and we can apply our procedure starting from this constraint. As before, for $\overline{c} < 0$ we obtain $t = \pm v$, while for $\overline{c} > 0$ we obtain $t = \pm u$. Note that now there is an important new feature: because both u and v depend on the 'intermediate' coordinate s which involves in its definition the original momenta, the time is extrinsic, that is, $t = t(q^i, p_i)$ (in the case $\overline{c} < 0$ an intrinsic time does not exist). However, in the case of interest for us, which is $\overline{c} > 0$, we have the same behaviour of t with Ω that we had in the absence of the antisymmetric field; t goes from $-\infty$ to 0 on the sheet $p_u > 0$ of the constraint surface and from 0 to ∞ on the other sheet, while $t \to 0$ for the singularity $\Omega \to -\infty$.

Though the problem of time in quantum cosmology is far from having been solved, we believe that at the level of the minisuperspace approximation much can be done towards a consistent formulation, both within the path integral formalism and in the usual canonical formalism. Here, within the context of the problem of non equivalent quantizations

yieldind from the same classical theory, we have discussed essentially three points: the correspondence between our path integral approach and a Schrödinger equation, the step of our deparametrization procedure where the choice takes place, and finally the possibility of avoiding the necessity of deciding between non equivalent quantizations by means of a suitable coordinate choice. Of course, the latter works for a limited class of homogeneous models, but, as we could see, it includes both string and relativistic, isotropic and some anisotropic, cosmologies; also, we have seen that we can even deal with models which do not admit an intrinsic time.

The possibility of working with an extrinsic time suggests an alternative formulation in the line of Ref. [21]. There, we avoided the intermediate coordinates $\tilde{q}^i$ by a straightforward procedure which led to a quantization in which the states were not characterized by the coordinates, but by the momenta; thus we obtained a path integral quantization with a good global phase time for the closed ($k = 1$) de Sitter universe. Working with an extrinsic time allowed to avoid the splitting of the formulation into two disjoint theories, even when an intrinsic time existed (see the case of the open ($k = -1$) de Sitter universe in the same paper); an attemp to generalize this analysis could be an interesting line of work to be followed.

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